Reduction of multisymplectic manifolds

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Good Morning SFARS
7 June 2021

Outline

1. Background
2. Symplectic reduction
3. Multisymplectic reduction

Based on:

B., Reduction of multisymplectic manifolds, *Lett. Math. Phys.*, 2021

1. Background

Preliminary Remarks

1. Reduction is a procedure that takes a space and returns a "smaller" space

$$(M, X) \xmapsto{\text{restrict \& quotient}} (M_{\mathrm{red}}, X_{\mathrm{red}})$$

2. The classic reduction result is the *Marsden–Weinstein–Meyer symplectic reduction theorem*.
3. Reduction theorems are rife in adjacent fields:
   contact, cosymplectic, polysymplectic, Poisson, Courant, quasi-Hamiltonian, ...

The Problem of Multisymplectic Reduction

> *Reduction theory is by no means completed. . . . Only a few instances and examples of multisymplectic reduction are really well understood. . . so one can expect to see more activity in this area as well.*

— J. Marsden and A. Weinstein, 2001, *Comments on the history, theory, and applications of symplectic reduction*

> *One of the most interesting problems in multisymplectic geometry is how to extend the well-known Marsden–Weinstein reduction scheme for symplectic manifolds to the case of multisymplectic structures.*

— M. de León, 2018, *Review of "Remarks on multisymplectic reduction" by Echeverría-Enríquez, Muñoz-Lecanda, and Román-Roy*

2. Symplectic Reduction

Conventions

- M and G are connected,
- $\xi, \zeta \in \mathfrak{g}$,
- $\lambda, \tau \in \mathfrak{g}^*$,
- for $\mu \in \Omega^*(M, \mathfrak{g}^*)$ and $\xi \in \mathfrak{g}$, write

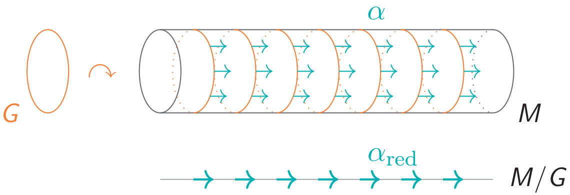
$$\mu_\xi := \langle \mu, \xi \rangle \in \Omega^*(M)$$

for the "ξth component" of μ.

The Action Descent Lemma

If

- $G \curvearrowright M$ free and proper,
- $\alpha \in \Omega^*(M)$ *invariant* and *horizontal* ($\iota_{\underline{\mathfrak{g}}}\alpha = 0$),



then

- $\exists!\, \alpha_{\mathrm{red}} \in \Omega^*(M/G)$ such that $\alpha = \pi^*\alpha_{\mathrm{red}}$,
- $\mathrm{d}\alpha = 0 \implies \mathrm{d}\alpha_{\mathrm{red}} = 0$.

Symplectic Manifolds

- M a smooth manifold.

Definition

A *symplectic structure* on M is a 2-form $\omega \in \Omega^2(M)$ which is

1. *nondegenerate:* $\forall X \in TM \backslash \{0\} : \iota_X \omega \neq 0$,

$$TM \xrightarrow{\sim} T^*M$$

2. *closed:* $\mathrm{d}\omega = 0$.

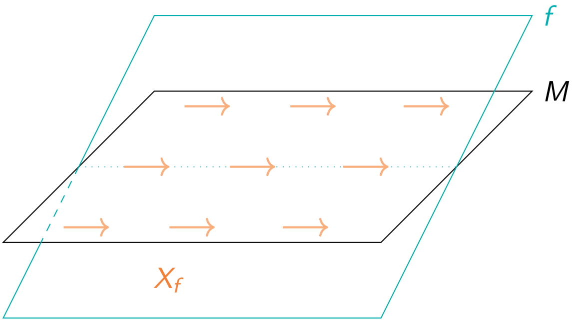
In his *Lectures on the Orbit Method*, A. Kirillov identifies three primary sources of examples:

- phase spaces T^*Q
- complex algebraic manifolds
- coadjoint orbits $\mathcal{O}_\lambda \subseteq \mathfrak{g}^*$

Symplectic Hamiltonian Dynamics

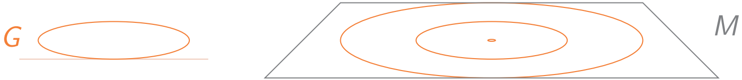
observables	$\longrightarrow$	symmetries
$C^\infty(M) \ni f$		$X \in \mathfrak{X}(M)\ , \quad \mathcal{L}_X\omega = 0$

$$\mathrm{d}f = \iota_{X_f}\omega$$

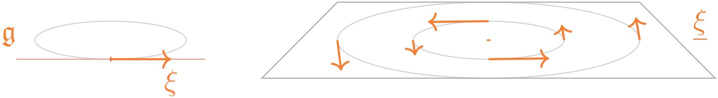


Symplectic Hamiltonian Actions

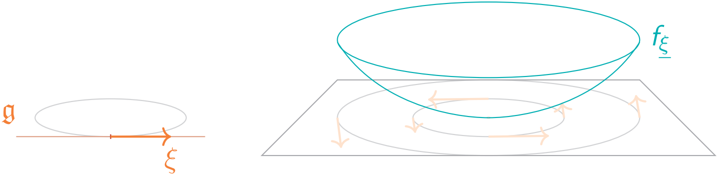
To specify a symplectic action $G \curvearrowright M$ ...



we could describe the induced map $\xi \mapsto \underline{\xi}$ ...



or an assignment of Hamiltonian functions $\xi \mapsto f_{\underline{\xi}}$.

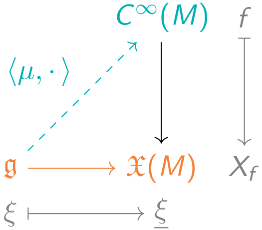


When this is possible,* the action is called **Hamiltonian**.

*and $\xi \mapsto f_{\underline{\xi}}$ is a homomorphism of Lie algebras

Symplectic Hamiltonian G-Spaces

moment map: Describe $G \curvearrowright M$ *in terms of* $C^\infty(M) \to \mathfrak{X}(M)$.



as Lie algebras

i.e., μ_ξ generates $\underline{\xi}$

moment map: $\mu : M \to \mathfrak{g}^*$

Hamiltonian G-space: (M, ω, G, μ)

Symplectic Reduction

Two inputs:

1. Hamiltonian G-space (M, ω, G, μ)
2. parameter $\lambda \in \mathfrak{g}^*$

The *reduced space* is $M_\lambda := \mu^{-1}(\lambda)/G_\lambda$.

Theorem (Marsden–Weinstein '74, Meyer '73)

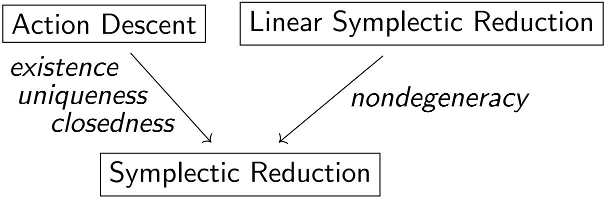
If $\mu^{-1}(\lambda) \subseteq M$ is smooth and $G_\lambda \curvearrowright \mu^{-1}(\lambda)$ is free and proper, then there is a unique symplectic form $\omega_\lambda \in \Omega^2(M_\lambda)$ *such that $\pi^*\omega_\lambda = i^*\omega$.*

restrict to $\{\mu = \lambda\}$ — $i^*\omega$, $\pi^*\omega_\lambda$: $\mu^{-1}(\lambda) \xhookrightarrow{i} M$ — ω

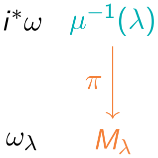
quotient by G_λ — $\pi: \mu^{-1}(\lambda) \to M_\lambda$

ω_λ — M_λ

Symplectic Reduction — Proof



1. Apply the *Action Descent Lemma* to $G_\lambda \curvearrowright \mu^{-1}(\lambda)$ and $i^*\omega$.



2. Use *Linear Symplectic Reduction* to conclude that ω_λ is nondegenerate.

Heuristic Approach to Reduction

1. describe $G \curvearrowright M$ in terms of ω
   moment map μ
2. identify a distinguished reduced space
   reduction at $\mu = 0$
3. use the ambiguity in 1. to obtain a family of reduced spaces
   reduction at $\mu - \lambda = 0$,
   i.e. reduction at $\mu = \lambda$

Note: If $G_\lambda \neq G$, then $\mu - \lambda : M \to \mathfrak{g}^*$ is not a moment map for either $G \curvearrowright M$ or $G_\lambda \curvearrowright M$.

3. Multisymplectic Reduction

Multisymplectic Manifolds

- M a smooth manifold.

Definition

A *k-plectic structure* on M is a $(k+1)$-form $\omega \in \Omega^{k+1}(M)$ which is

1. *nondegenerate:* $\forall X \in TM \backslash \{0\} \ : \ \iota_X \omega \neq 0$,

$$TM \hookrightarrow \Lambda^k T^* M$$

2. *closed:* $\mathrm{d}\omega = 0$,

- $\{$1-plectic structures$\} = \{$symplectic structures$\}$
- Multisymplectic geometry arises as a natural framework for classical field theories.

Multisymplectic Hamiltonian Dynamics

observables	$\longrightarrow$	symmetries
~~$C^\infty(M) \ni f$~~		$X \in \mathfrak{X}(M)\,, \quad \mathcal{L}_X\omega = 0$
$\Omega_H^{k-1}(M) \ni \alpha$		

$$\mathrm{d}\alpha = \iota_{X_\alpha}\omega$$

Hamiltonian vector fields are indeed multisymplectic symmetries:

$$\begin{aligned}\mathcal{L}_{X_\alpha}\omega &= \mathrm{d}\iota_{X_\alpha}\omega + \iota_{X_\alpha}\mathrm{d}\omega \\ &= \mathrm{d}\mathrm{d}\alpha \\ &= 0\end{aligned}$$

Two brackets on $\Omega_H^{k-1}(M)$

1. $\{\alpha, \beta\} = \mathcal{L}_{X_\alpha}\beta$ ← ***Leibniz bracket*** (Jacobi, but not antisymmetric)
2. $\{\alpha, \beta\}' = \iota_{X_\alpha}\mathrm{d}\beta = \iota_{X_\alpha}\iota_{X_\beta}\omega$

Equal up to a coboundary:

$$\mathcal{L}_{X_\alpha}\beta = \iota_{X_\alpha}\mathrm{d}\beta + \mathrm{d}\iota_{X_\alpha}\beta$$

$\alpha \mapsto X_\alpha$ is a homomorphism of Leibniz algebras:

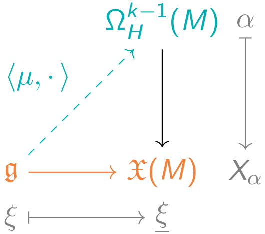
$$\mathrm{d}\{\alpha, \beta\} = \mathrm{d}\mathcal{L}_{X_\alpha}\beta = \mathcal{L}_{X_\alpha}\iota_{X_\beta}\omega = \iota_{[X_\alpha, X_\beta]}\omega$$

$$\implies \quad X_{\{\alpha,\beta\}} = [X_\alpha, X_\beta]$$

Multisymplectic Hamiltonian G-Spaces

moment map: Describe $G \curvearrowright M$ *in terms of* ~~$C^\infty(M)$~~ $\to \mathfrak{X}(M)$.

$\Omega_H^{k-1}(M)$



as Leibniz algebras

i.e., μ_ξ generates $\underline{\xi}$

moment map: $\mu \in \Omega^{k-1}(M, \mathfrak{g}^*)$

Hamiltonian G-space: (M, ω, G, μ)

The Moment Map Conditions

- $G \curvearrowright M$
- $\mu \in \Omega^{k-1}(M, \mathfrak{g}^*)$

1. Hamiltonian condition:

$$\mathrm{d}\mu_\xi = \iota_\xi \omega$$

2. Leibniz condition:

$$\mu_{[\xi,\zeta]} = \{\mu_\xi, \mu_\zeta\}$$

equivalently,

$$\mathcal{L}_\xi \mu_\zeta = \mu_{[\xi,\zeta]}$$

The Space of Moment Maps

- (M, ω, G, μ) Hamiltonian G-space
- $\phi \in \Omega^{k-1}(M, \mathfrak{g}^*)$

Question: When is $\mu + \phi$ a moment map?

- $\mathrm{d}\phi = 0$, since

$$\mathrm{d}(\mu + \phi)_\xi = \iota_\xi \omega \iff \mathrm{d}\phi_\xi = 0.$$

- $\mathcal{L}_\xi \phi_\zeta = \phi_{[\xi,\zeta]}$, as

$$\mathcal{L}_\xi (\mu + \phi) = (\mu + \phi)_{[\xi,\zeta]} \iff \mathcal{L}_\xi \phi = \phi_{[\xi,\zeta]}.$$

i.e. ϕ is a moment map for the trivial action $G \curvearrowright M$.

The space of moment maps is an affine space modeled on $\{\phi \in \Omega^{k-1}(M, \mathfrak{g}^*) \mid \mathrm{d}\phi = 0,\ G_\phi = G\}$.

The Leibniz Condition and the Induced Action on Forms

- $\phi \in \Omega^*(M, \mathfrak{g}^*)$
- $\xi \in \mathfrak{g}$

$$\forall \zeta \in \mathfrak{g} : \quad \mathcal{L}_\xi \phi_\zeta = \phi_{[\xi,\zeta]} \iff \forall \zeta \in \mathfrak{g} : \quad \begin{aligned} 0 &= \mathcal{L}_\xi \phi_\zeta - \phi_{[\xi,\zeta]} \\ &= \mathcal{L}_\xi \phi_\zeta + \langle \mathrm{ad}^*_\xi \phi, \zeta \rangle \\ &= \langle \mathcal{L}_\xi \phi + \mathrm{ad}^*_\xi \phi, \zeta \rangle \end{aligned}$$

$$\iff \quad 0 = \left(\mathcal{L}_\xi + \mathrm{ad}^*_\xi \right) \phi$$

$$\iff \quad \xi \in \mathfrak{g}_\phi$$

in terms of the induced action $G \curvearrowright \Omega^*(M, \mathfrak{g}^*)$. Thus,

$$\forall \xi, \zeta \in \mathfrak{g} : \quad \mathcal{L}_\xi \phi_\zeta = \phi_{[\xi,\zeta]} \iff G \cdot \phi = \phi$$

Level Sets of the Moment Map

Rather than:

- family of moment maps $\{\mu - \phi \,|\, \mathrm{d}\phi = 0,\ G_\phi = G\}$
- reduction at $\mu - \phi = 0$

We instead consider:

- fixed moment map μ
- family of levels $\{\phi \,|\, \mathrm{d}\phi = 0,\ \cancel{G_\phi = G}\}$
- reduction at $\mu = \phi$

ϕ-level set:

$$\mu^{-1}(\phi) := \{\mu = \phi\}$$

Multisymplectic Reduction

Two inputs:

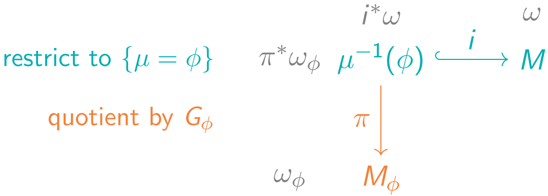
1. Hamiltonian G-space (M, ω, G, μ)
2. parameter $\phi \in \Omega^{k-1}(M, \mathfrak{g}^*)$ with $\mathrm{d}\phi = 0$

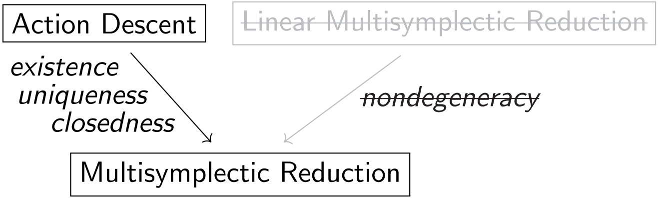
Define the *reduced space*,

- $M_\phi := \mu^{-1}(\phi)/G_\phi$

Theorem (B. '20)

If $\mu^{-1}(\phi) \subseteq M$ is smooth and $G_\phi \curvearrowright \mu^{-1}(\phi)$ is free and proper, then there is a unique closed $(k+1)$-form ω_ϕ *on M_ϕ such that $\pi^*\omega_\phi = i^*\omega$.*





1. Apply the *Action Descent Lemma* to $G_\phi \curvearrowright \mu^{-1}(\phi)$ and $i^*\omega$.

$$\begin{array}{cc} i^*\omega & \mu^{-1}(\phi) \\ & \downarrow \pi \\ \omega_\phi & M_\phi \end{array}$$

2. ~~Use *Linear Multisymplectic Reduction* to conclude that ω_ϕ is nondegenerate.~~

Multisymplectic Reduction — Proof Outline

Two steps:

1. $G_\phi \curvearrowright M$ preserves $\mu^{-1}(\phi)$,
2. $i^*\omega$ is invariant and horizontal.

Multisymplectic Reduction — Proof (Step 1)

1. $G_\phi \curvearrowright M$ preserves $\mu^{-1}(\phi)$.

- $\mu^{-1}(\phi) = \{\mu - \phi = 0\}$
- $\forall\, \xi, \zeta \in \mathfrak{g}_\phi$,

$$\begin{aligned} \mathcal{L}_\xi (\mu - \phi)_\zeta &= (\mu - \phi)_{[\xi,\zeta]}, && \text{by the Leibniz condition,} \\ &= 0 && \text{on } \mu^{-1}(\phi). \end{aligned}$$

Multisymplectic Reduction — Proof (Step 2)

2. $i^*\omega$ is invariant and horizontal.

- **invariant:** Hamiltonian actions are multisymplectic.
- **horizontal:** For $\xi \in \mathfrak{g}_\phi$,

$$\begin{aligned} \iota_\xi i^*\omega &= i^*\iota_\xi\omega, && \text{since } G_\phi \text{ preserves } \mu^{-1}(\phi), \\ &= i^*\mathrm{d}\mu_\xi, && \text{by the Hamiltonian condition,} \\ &= i^*\mathrm{d}\phi_\xi, && \text{since } \mu = \phi \text{ on } \mu^{-1}(\phi), \\ &= 0, && \text{as } \phi \text{ is closed.} \end{aligned}$$

□

Thank you!